Last time: More basic logic.
Logical equivalence.

Addendum: Add to your table of Equivalences:

Idempotence: $P \wedge P \equiv P$, $P \vee P \equiv P$

Remark: There is some redundancy in our table of equivalences...

Ex: Is $\neg(P \to Q)$ logically equivalent to $(\neg P) \to (\neg Q)$?

Sol:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg(P \to Q)$	$(\neg P) \to (\neg Q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	F	T

Therefore $\neg(P \to Q) \not\equiv (\neg P) \to (\neg Q)$ by definition ☒

Algebra of Logical Equivalence

IDEA: Build from the table of equivalences to find new equivalences.

Ex: We use the basic equivalences to derive

$$\neg(P \to Q) \equiv P \wedge (\neg Q).$$

In particular

$$\neg(P \to Q) \equiv \neg((\neg P) \vee Q) \quad \text{(Material implication)}$$

Ex: We show $\neg(P \to Q) \equiv P \wedge (\neg Q)$ via string of equivs.

$\neg(P \to Q)$

$\equiv \neg((\neg P) \vee Q)$ (material implication)

$\equiv (\neg(\neg P)) \wedge (\neg Q)$ (De Morgan's Law)

$\equiv P \wedge (\neg Q)$ (Double Negation) ☑

Ex: Show $(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$ using the algebra of equivalences.

Sol:

$(P \to Q) \vee (P \to R)$ ↓

$\equiv ((\neg P) \vee Q) \vee (P \to R)$ (Material Implication)

$\equiv ((\neg P) \vee Q) \vee ((\neg P) \vee R)$ (Material Implication)

$\equiv (\neg P) \vee (Q \vee ((\neg P) \vee R))$ (Associativity)

$\equiv (\neg P) \vee ((Q \vee (\neg P)) \vee R)$ (Assoc.)

$\equiv (\neg P) \vee (((\neg P) \vee Q) \vee R)$ (commutativity)

$\equiv (\neg P) \vee ((\neg P) \vee (Q \vee R))$ (Assoc)

$\equiv ((\neg P) \vee (\neg P)) \vee (Q \vee R)$ (Assoc)

$\equiv (\neg P) \vee (Q \vee R)$ (Idempotence)

$\equiv P \to (Q \vee R)$ (Mat. Imp.) ☑

Crucial Observation: This is only good for proving equivalence...

by Soon: we'll learn how to prove implications...

NB: Proving non-equivalence using statement algebra requires one more piece of knowledge: Disjunctive Normal Forms...

Ex: Build the disjunctive normal form of $P \to Q$ (from its truth table):

P	Q	P→Q	
T	T	T	← $P \wedge Q$ ← true only in 1st row
T	F	F	← False, so ignore...
F	T	T	← $(\neg P) \wedge Q$ ← true only in 3rd row
F	F	T	← $(\neg P) \wedge (\neg Q)$ true only in 4th row

So the disjunctive normal form is the OR- of these statements

i.e. $P \to Q \equiv (P \wedge Q) \vee ((\neg P) \wedge Q) \vee ((\neg P) \wedge (\neg Q))$.

Exercise: Verify this equivalence via truth table. ☐

Prop: There is a unique disjunctive normal form for any given statement in the propositional logic.

The DNF is constructed (from the truth table) by

① Identifying all "true rows" of the table.

② Constructing the conjunction associated to each "true row"

③ Taking the "total or" of these "true row conjunctions"

Prop: Our logical equivalence table is "logically complete".

I.E. Any two equivalent (propositional) statements can be connected via a string of basic equivalences.

NB: Thus we can compare DNF's (= "disjunctive normal form") computed via basic equivalences to decide statements are (non) equivalent.

Arguments and Argument Forms

An argument is a sequence of (propositional) deductions.

An argument is ...

- valid when assigning true to all premises yields a true conclusion.
- invalid otherwise.

A premise is an assumed/given statement. The conclusion of an argument is its final deduced statement.

Ex: [If the dragon eats meat, then the dragon is not vegan.] [The dragon does eat meat.] [Therefore, the dragon is not vegan.]

premises. conclusion.

In the propositional world, this translates to:

$$\left(\begin{cases} M : \text{the dragon eats meat} \\ V : \text{the dragon is vegan.} \end{cases} \leftarrow \text{dictionary} \right)$$

$$[M \to (\neg V),\ M,\ \therefore \neg V] \leftarrow \text{argument form.}$$

Notation: If an argument form has premises $P_1, P_2, \dots, P_n$ and conclusion q, we denote the form by

$$P_1, P_2, \dots, P_n \ \therefore q$$

Test for validity: Argument form $P_1, P_2, \dots, P_n \ \therefore q$ is valid precisely when $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \longrightarrow q$ is a tautology.

Ex: Analyzing the previous argument $(M \to (\neg V),\ M \ \therefore \neg V)$

M	V	$\neg V$	$M \to (\neg V)$	$(M \to (\neg V)) \wedge M$	$((M \to (\neg V)) \wedge M) \to (\neg V)$
T	T	F	F	F	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T

$\therefore$ this argument form is valid by our validity test.

Q: Why $((M \to (\neg V)) \wedge M) \to \neg V$ in the last column?

$M \to (\neg V)$, M $\therefore \neg V$ is <u>valid</u> precisely when
P_1 P_2 q

$(P_1 \wedge P_2) \to q$ is tautological

Basic Valid Argument Forms

Name	Argument Form
Modus Ponens	$P \to Q$, P $\therefore Q$
Modus Tollens	$P \to Q$, $\neg Q$ $\therefore \neg P$
Disjunctive Syllogism	$P \vee Q$, $\neg P$ $\therefore Q$
Hypothetical Syllogism	$P \to Q$, $Q \to R$ $\therefore P \to R$
Dilemma	$P \vee Q$, $P \to R$, $Q \to R$ $\therefore R$
Reductio Ad Absurdum	$(\neg P) \to (Q \wedge (\neg Q))$ $\therefore P$
Conjunctive Simplification	$P \wedge Q$ $\therefore P$
Conjunctive Addition	P, Q $\therefore P \wedge Q$
Disjunctive Addition	P $\therefore P \vee Q$
Biconditional Syllogism	$P \leftrightarrow Q$, $Q \leftrightarrow R$ $\therefore P \leftrightarrow R$

<u>Exercise:</u> Verify the validity of each argument above.